

Numerical methods for optimal hedging portfolios

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1. Introduction

- **Market Model:** We consider an n -factor Markovian market with state variables $(S^1, \dots, S^d, Y^1, \dots, Y^{n-d})$ where S_t is the \mathbb{R}^d -valued process which describes the discounted prices of traded assets and Y_t is the \mathbb{R}^{n-d} -valued process corresponding to the values of nontraded quantities such as stochastic volatilities which may or may not be observed directly.

For example, we treat a two factor stochastic volatility model

$$\begin{aligned} dS_t &= S_t[(\mu(t, Y_t) - r)dt + \sigma(t, Y_t)dW_t^1] \\ dY_t &= a(t, Y_t)dt + b(t, Y_t)[\rho dW_t^1 + \sqrt{1 - \rho^2}dW_t^2] \end{aligned} \quad (1)$$

with initial values $S_0, Y_0 \geq 0$, for deterministic functions μ, a, b and independent one dimensional P -Brownian motions W_t^1 and W_t^2 with constant correlation $|\rho| \leq 1$.

- **Optimal hedging portfolio:** the strategy followed by an investor who, when faced with a (discounted) financial liability B maturing at a future time T , tries to solve the stochastic control problem

$$u(x) = \sup_{H \in \mathcal{A}} E [U (X_T - B)], \quad (2)$$

where $X_t = x + (H \cdot S)_t$ is the discounted terminal wealth obtained when investing according to the **self financing** portfolio $H_t = (H_t^1, \dots, H_t^d)$ and the (discounted) liability B is assumed to be a random variable of the form $B = B(S_T, Y_T)$.

- **Utility function:** $U(x) = -\frac{e^{-\gamma x}}{\gamma}$, where $\gamma > 0$ is the risk aversion parameter.

2. Utility based pricing

For such Markovian markets we can embed the optimal hedging problem (2) into the larger class of optimization problems defined by

$$u(t, x, s, y) = \sup_{H \in \mathcal{A}_t} E_{t,s,y}[U(X_T - B(S_T, Y_T)) | X_t = x], \quad (3)$$

for $t \in (0, T)$, where $x \in \mathbb{R}$ denotes some arbitrary level of wealth, \mathcal{A}_t denotes admissible portfolios starting at time t and $E_{t,s,y}[\cdot]$ denotes expectation with respect to the joint probability law at time t of the processes S_u, Y_u , for $u \geq t$, with initial condition $S_t = s$ and $Y_t = y$.

Suppose that (3) has an optimizer H_t^B , that is, assume that

$$u(t, x, s, y) = E_{t,s,y}[U(x + (H^B \cdot S)_t^T - B(S_T, Y_T))],$$

Define the **certainty equivalent** for the claim B at time t as the process $c_t^B = c^B(t, x, s, y)$ satisfying the equation

$$U(x - c_t^B) = E_{t,s,y}[U(x + (H^B \cdot S)_t^T - B(S_T, Y_T))]. \quad (4)$$

If we set $B = 0$, then the optimal hedging problem becomes the Merton optimal investment problem and we denote the certainty equivalent by $c_t^0 = c^0(t, x, s, y)$.

The **indifference price** for the claim B is defined to be solution $\pi^B = \pi^B(t, x, s, y)$ to the equation

$$\sup_{H \in \mathcal{A}_t} E_{t,s,y}[U(x + (H \cdot S)_t^T)] = \sup_{H \in \mathcal{A}_t} E_{t,s,y}[U(x + \pi^B + (H \cdot S)_t^T - B(S_T, Y_T))]. \quad (5)$$

From the definition of the certainty equivalent, we see that this equation is equivalent to

$$U(x - c_t^0) = U(x + \pi^B - c_t^B), \quad (6)$$

so that the indifference price is given by

$$\pi^B(t, x, s, y) = c^B(t, x, s, y) - c^0(t, x, s, y). \quad (7)$$

3. Discrete time hedging

We now consider portfolio processes of the form

$$H_t = \sum_{k=1}^K H_k \mathbf{1}_{(t_{k-1}, t_k]}(t) \quad (8)$$

where each H_k is an \mathbb{R}^d -valued \mathcal{F}_{k-1} random variable. We take the discrete time partition of the interval $[0, T]$ to be of the form

$$t_0 = 0 < t_1 = \frac{T}{K} < \dots < t_k = \frac{kT}{K} \dots < t_K = T$$

and use the notation $S_j := S_{t_j}$ for discrete time stochastic processes.

The discounted wealth for self-financing portfolios is

$$X_j = x + (H \cdot S)_j, \quad (9)$$

with the notation $(H \cdot S)_k^j := (H \cdot S)_j - (H \cdot S)_k$, where

$$(H \cdot S)_j := \sum_{k=1}^j H_k \Delta S_k \quad (10)$$

and $\Delta S_k := S_k - S_{k-1}$.

Now the **dynamic programming problem** for the optimal hedge falls into K subproblems

$$u_{k-1}(x) = \sup_{H_k \in \mathcal{F}_{k-1}} E_{k-1}[u_k(x + H_k \Delta S_k)], \quad (11)$$

for $k = K, K - 1, \dots, 1$, with $u_K(x) = U(x - B)$. Similarly, the certainty equivalent value process $c_k^B(x)$ is defined iteratively by

$$U(x - c_{k-1}^B(x)) = \sup_{H_k \in \mathcal{F}_{k-1}} E_{k-1}[U(x + H_k \Delta S_k - c_k^B(x + H_k \Delta S_k))] \quad (12)$$

with $c_K^B(x)$ taken equal to the terminal discounted claim B .

In our **Markovian setting** and with an **exponential utility**, the solution of (11) and (12) as well as the optimal allocation H^B have the form wealth independent form

$$u_k = g_k(S_k, Y_k) \quad (13)$$

$$c_k^B = c_k(S_k, Y_k) \quad (14)$$

$$H_{k+1}^B = h_{k+1}(S_k, Y_k) \quad (15)$$

for (deterministic) Borel scalar functions $\{g_k, c_k\}_{k=0}^{K-1}$ and \mathbb{R}^d -valued functions $\{h_{k+1}\}_{k=0}^{K-1}$ on the state space \mathcal{S} .

4. The exponential utility allocation algorithm

We want an algorithm which will generate an approximate trading rule, based on a data set

$$\{(S_k^i, Y_k^i)\}_{i=1, \dots, N; k=0, \dots, K}$$

where $(S_k^i, Y_k^i) \in \mathbb{R}^n$ denotes the state of the i th sample path at time $t_k = kT/K$ for the processes (S_t, Y_t) . In the special case of an exponential utility, the theoretical optimal rule

$$H_{k+1}^B = h_k(S_k^i, Y_k^i)$$

in (15) depends only on the directly observed data $\{S_k^i, Y_k^i\}$ and is independent of the wealth X_k^i . For this reason our algorithm is at this point restricted to exponential utility functions, and we take $\gamma = 1$ for simplicity.

1. Step $k = K$: The final optimal allocation is the \mathcal{F}_{K-1} -random variable H_K^B which solves

$$\min_{H_K \in \mathcal{F}_{K-1}} E[\exp(-H \cdot \Delta S_K + B)]. \quad (16)$$

Since the solution is known to be given by $H_K^B = h_K(S_{K-1}, Y_{K-1})$ for some deterministic function $h_K \in \mathcal{B}(\mathcal{S})$ (the set of Borel functions on \mathcal{S}), we write this as

$$\min_{h \in \mathcal{B}(\mathcal{S})} E[\exp(-h(S_{K-1}, Y_{K-1}) \cdot \Delta S_K + B)]. \quad (17)$$

On a finite set of data, we can pick an R -dimensional subspace $\mathcal{R}(\mathcal{S}) \subset \mathcal{B}(\mathcal{S})$ of functions on \mathcal{S} and attempt to “learn” a suboptimal solution

$$\arg \min_{h \in \mathcal{R}(\mathcal{S})} E[\exp(-h(S_{K-1}, Y_{K-1}) \cdot \Delta S_K + B)].$$

By the central limit theorem, the expectation above can be approximated by the finite sample estimate

$$\Psi_K(h) = \frac{1}{N} \sum_{i=1}^N \exp \left(-h(S_{K-1}^i, Y_{K-1}^i) \cdot \Delta S_K^i + B(S_K^i, Y_K^i) \right) \quad (18)$$

This leads to the estimator $h_K^{\mathcal{R}}$ based on $\{S_k^i, Y_k^i\}$ and the choice of subspace \mathcal{R} defined by

$$h_K^{\mathcal{R}} = \arg \min_{h \in \mathcal{R}(S)} \Psi_K(h) \quad (19)$$

2. Inductive step for $k = K - 1, \dots, 2$: The estimate $h_k^{\mathcal{R}}$ of the optimal rule h_k , for the intermediate time steps $2 \leq k < K - 1$ is determined inductively given the estimates $h_{k+1}^{\mathcal{R}}, \dots, h_K^{\mathcal{R}}$. It is defined to be

$$h_k^{\mathcal{R}} = \arg \min_{h \in \mathcal{R}(S)} \Psi_k(h; h_{k+1}^{\mathcal{R}}, \dots, h_K^{\mathcal{R}}) \quad (20)$$

where

$$\Psi_k(h) = \frac{1}{N} \sum_{i=1}^N \exp \left(-h(S_k^i, Y_k^i) \cdot \Delta S_{k+1}^i + c_k^i(h_{k+1}^{\mathcal{R}}, \dots, h_K^{\mathcal{R}}, S_K^i, Y_K^i) \right), \quad (21)$$

with

$$c_k^i(h_{k+1}^{\mathcal{R}}, \dots, h_K^{\mathcal{R}}, S_K^i, Y_K^i) = B(S_K^i, Y_K^i) - \sum_{j=k+1}^K h_j^{\mathcal{R}}(S_{j-1}^i, Y_{j-1}^i) \cdot \Delta S_j^i \quad (22)$$

3. Final step $k = 1$: This step is degenerate since the initial values (S_0, Y_0) are constant over the sample. Therefore we determine the optimal constant vector $h_1 \in \mathbb{R}^d$ by solving

$$h_1 = \arg \min_{h \in \mathbb{R}^d} \Psi_1(h; h_2^{\mathcal{R}}, \dots, h_K^{\mathcal{R}}) \quad (23)$$

Finally, the optimal value

$$\Psi_1 = \frac{1}{N} \sum_{i=1}^N \exp \left(-h_1(S_0, Y_0) - \sum_{j=2}^K h_j^{\mathcal{R}}(S_{j-1}^i, Y_{j-1}^i) \cdot \Delta S_j^i + B(S_K^i, Y_K^i) \right),$$

is an estimate of the quantity $\exp(c_0^B)$, where c_0^B is the certainty equivalent value of the claim B at time $t = 0$

5. Numerical results

Geometric Brownian motion

We start with a one dimensional complete market in order to test the algorithm against well known exact solutions. Consider a market where the stock price process, discounted by the constant interest rate r , satisfy

$$\frac{dS_t}{S_t} = (\mu - r)dt + \sigma dW, \quad (24)$$

where μ and $\sigma > 0$ are constants and W is a one-dimensional P -Brownian motion.

As it is well known, the unique equivalent martingale measure Q has density dQ/dP given by the stochastic exponential of the constant market price of risk $\lambda = (\mu - r)/\sigma$ and the Merton portfolio for this market is given by

$$\widehat{H}_t = \frac{\mu - r}{\gamma \sigma^2} \frac{1}{S_t}. \quad (25)$$

We can now compare the hedging portfolio “learned” by our algorithm with the “true” optimal hedging portfolio given

$$H_t^B = \widehat{H}_t + \mathcal{H}_t^B, \quad (26)$$

where \mathcal{H}_t^B is the Black–Scholes *delta hedging* portfolio replicating B . Similarly, the indifference prices calculated by the algorithm can be compared with the Black–Scholes price for the same claim.

We fix the parameters of the model at

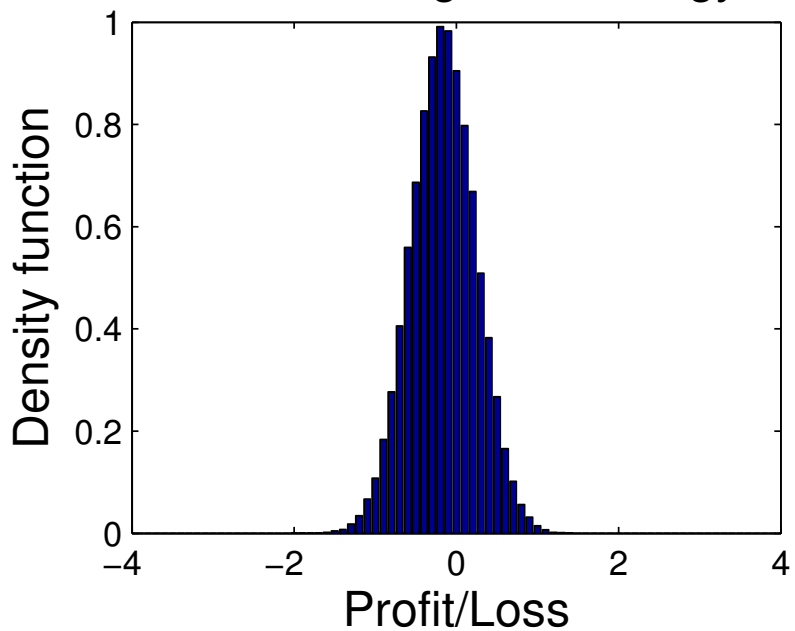
$$S_0 = 1, \quad \mu = 0.1, \quad \sigma = 0.2 \quad \text{and} \quad r = 0.02$$

over the period of one year $T = 1$ and discrete time intervals of $1/50$.

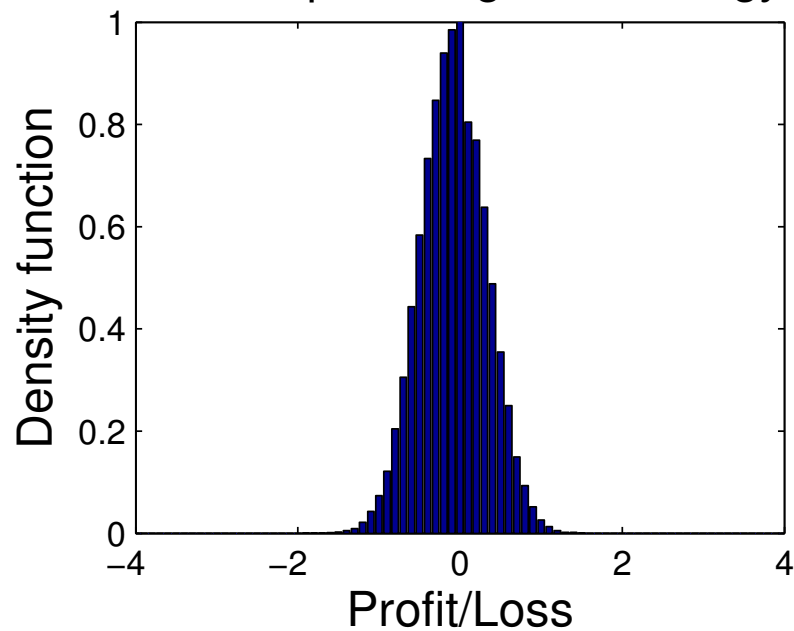
We apply the allocation algorithm with $N = 100000$ to two scenarios: (i) the Merton investment problem; and (ii) the hedging problem for the *writer* of a single written at-the-money European put. Then, for comparison to theory, we use the same Monte Carlo simulations, but rehedge weekly according to the theoretical formula (26).

As for the subspace $\mathcal{R}(\mathcal{S})$, we use the three dimensional space spanned by the functions $\{1, s, s^2\}$.

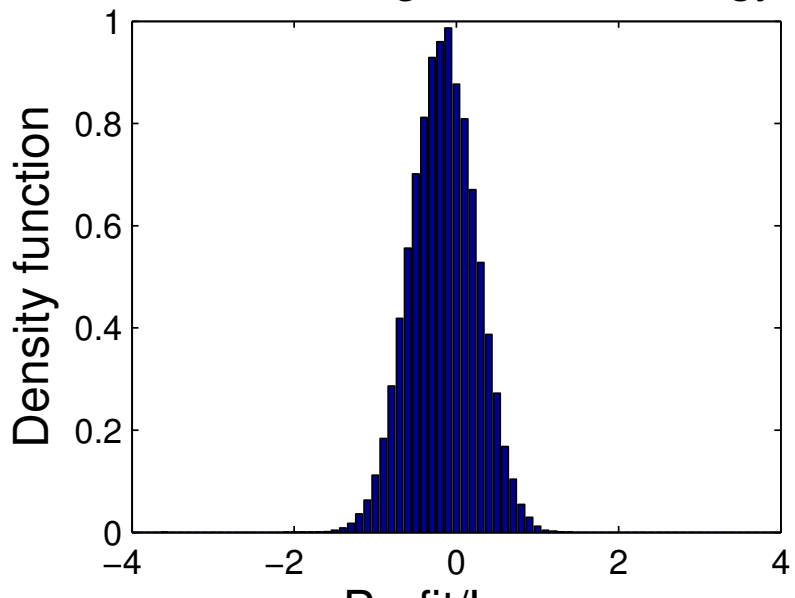
Merton using true strategy



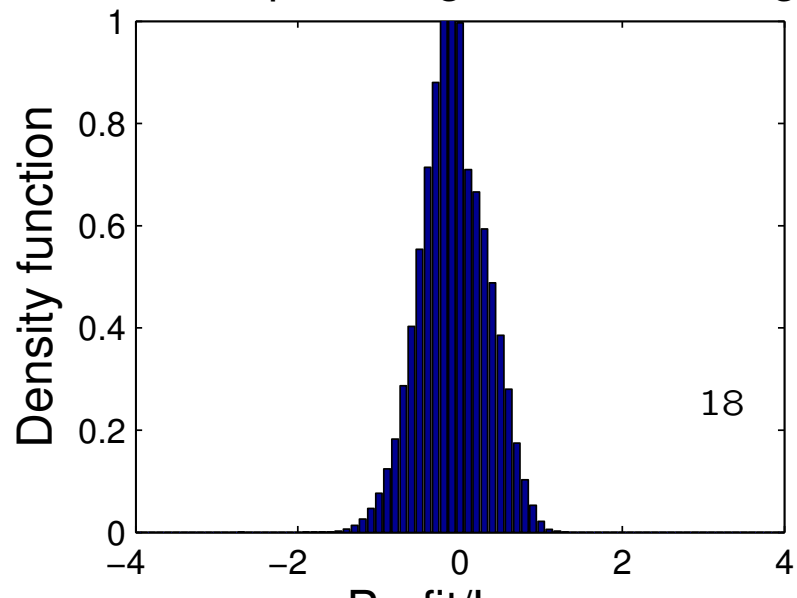
Written put using true strategy



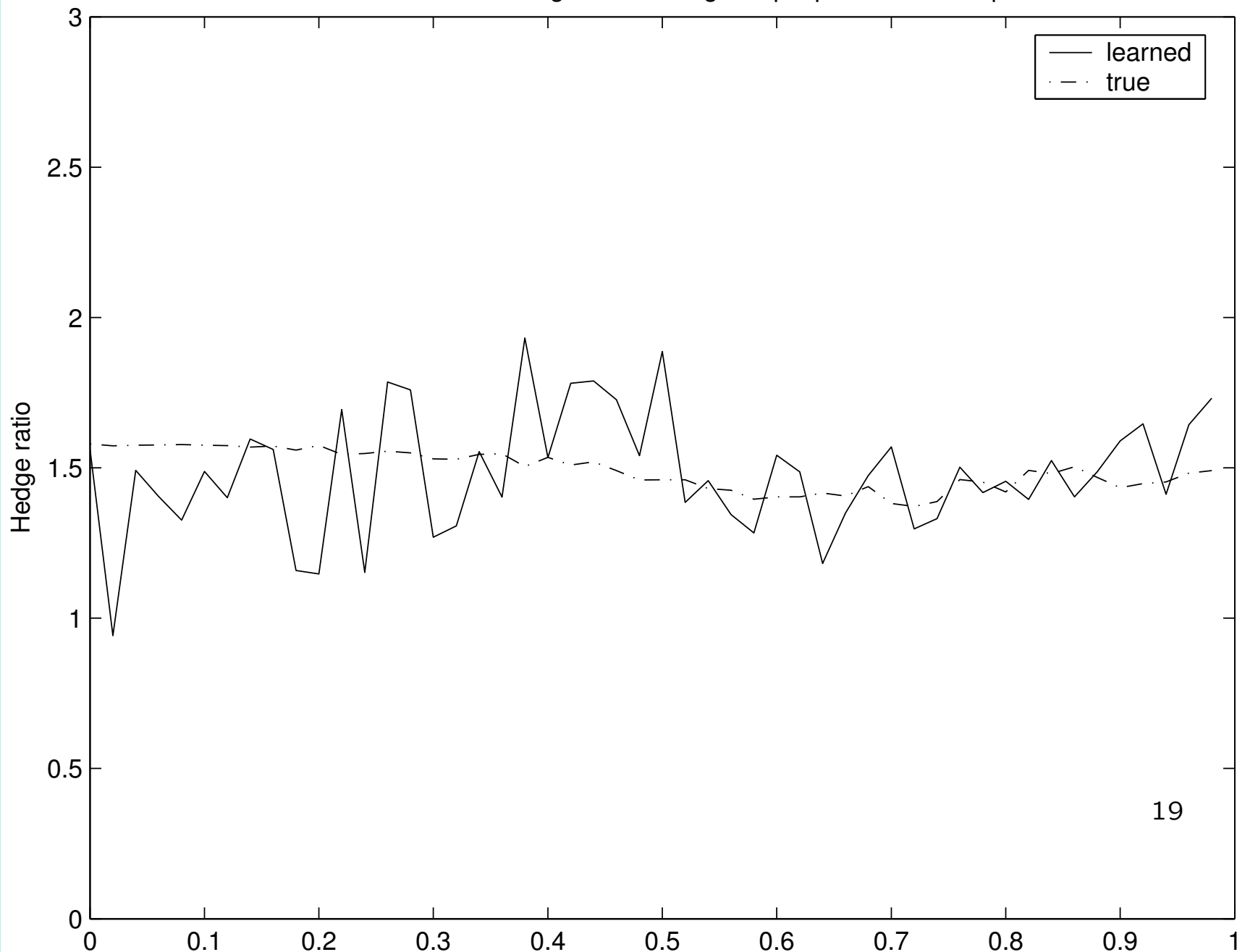
Merton using learned strategy



Written put using learned strategy



True and learned hedge ratios along sample path for written put



Reciprocal affine stochastic volatility models

We now take μ and r to be constants and $\sigma(t, Y_t) = \sqrt{Y_t}$ in (1). Define

$$R_t = R(t, Y_t) = \frac{(1 - \rho^2)(\mu - r)^2}{2Y_t}, \quad (27)$$

which we postulate to be a **CIR process**, that is

$$dR_t = \alpha(\kappa - R_t)dt + \beta\sqrt{R_t} \left[\rho dW_t^1 + \sqrt{1 - \rho^2} dW_t^2 \right], \quad (28)$$

for constants $\alpha, \kappa, \beta > 0$ with $4\alpha\kappa > \beta^2$.

We then obtain from the Itô formula that

$$a(t, Y_t) = \alpha Y_t + \frac{2(\beta^2 - \alpha\kappa)}{(1 - \rho^2)(\mu - r)^2} Y_t^2, \quad (29)$$

$$b(t, Y_t) = - \left(\frac{2}{1 - \rho^2} \right)^{1/2} \frac{\beta}{(\mu - r)} Y_t^{3/2}. \quad (30)$$

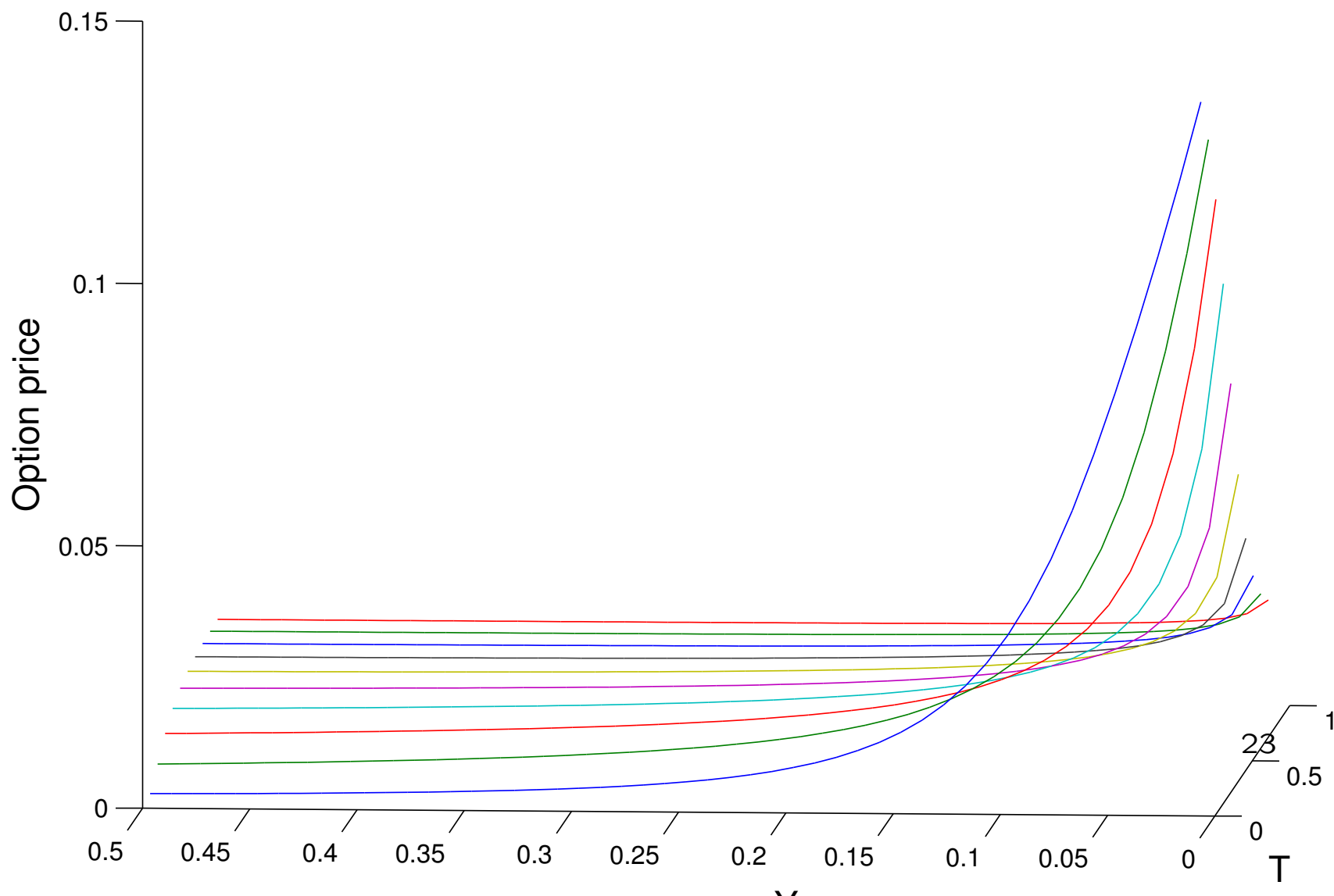
It follows from the *Hamilton–Jacobi–Bellman* equations associated with (3), that indifference prices and optimal hedging portfolios for **pure volatility claims** of the form $B = B(Y_T)$ can be explicitly computed using a Fourier transform technique. We use these as benchmarks for our more general Monte Carlo algorithm.

We fix the model parameters at reasonable values:

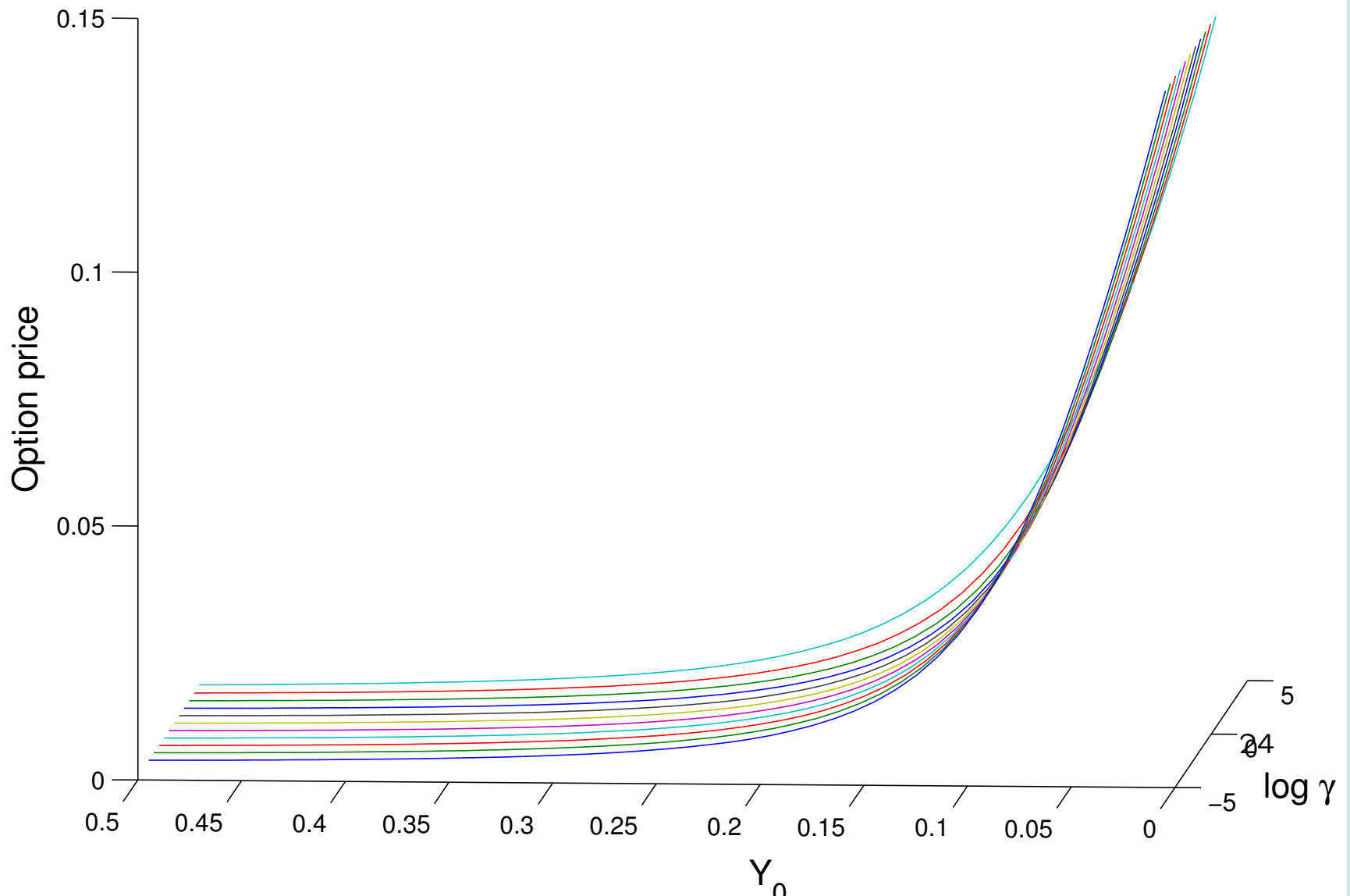
$$\begin{aligned}\alpha &= 5, & \beta &= 0.04, & \kappa &= 0.001, \\ \mu &= 0.04, & r &= 0.02, & \rho &= 0.5\end{aligned}$$

and initial squared volatility ranging in the interval $[0, 0.5]$. With these parameters the squared volatility process has a mean reversion time of approximately two months and an equilibrium distribution with expected value approximately 40%. We calculate the price of a put option on volatility with payoff $(0.15 - \sigma_T^2)^+$. When not mentioned the risk aversion parameter is set to $\gamma = 1$.

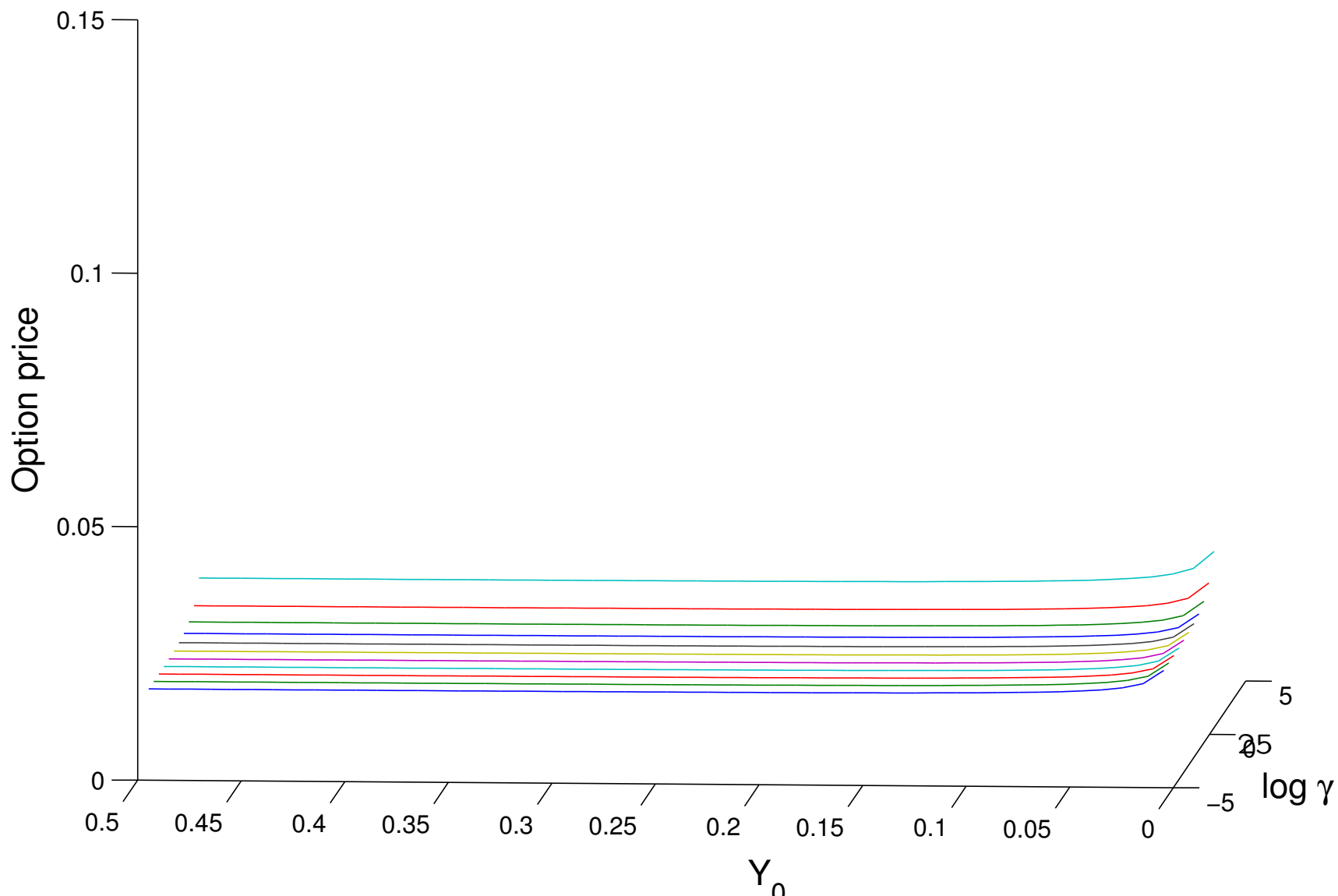
Volatility put versus time to maturity and Y_0



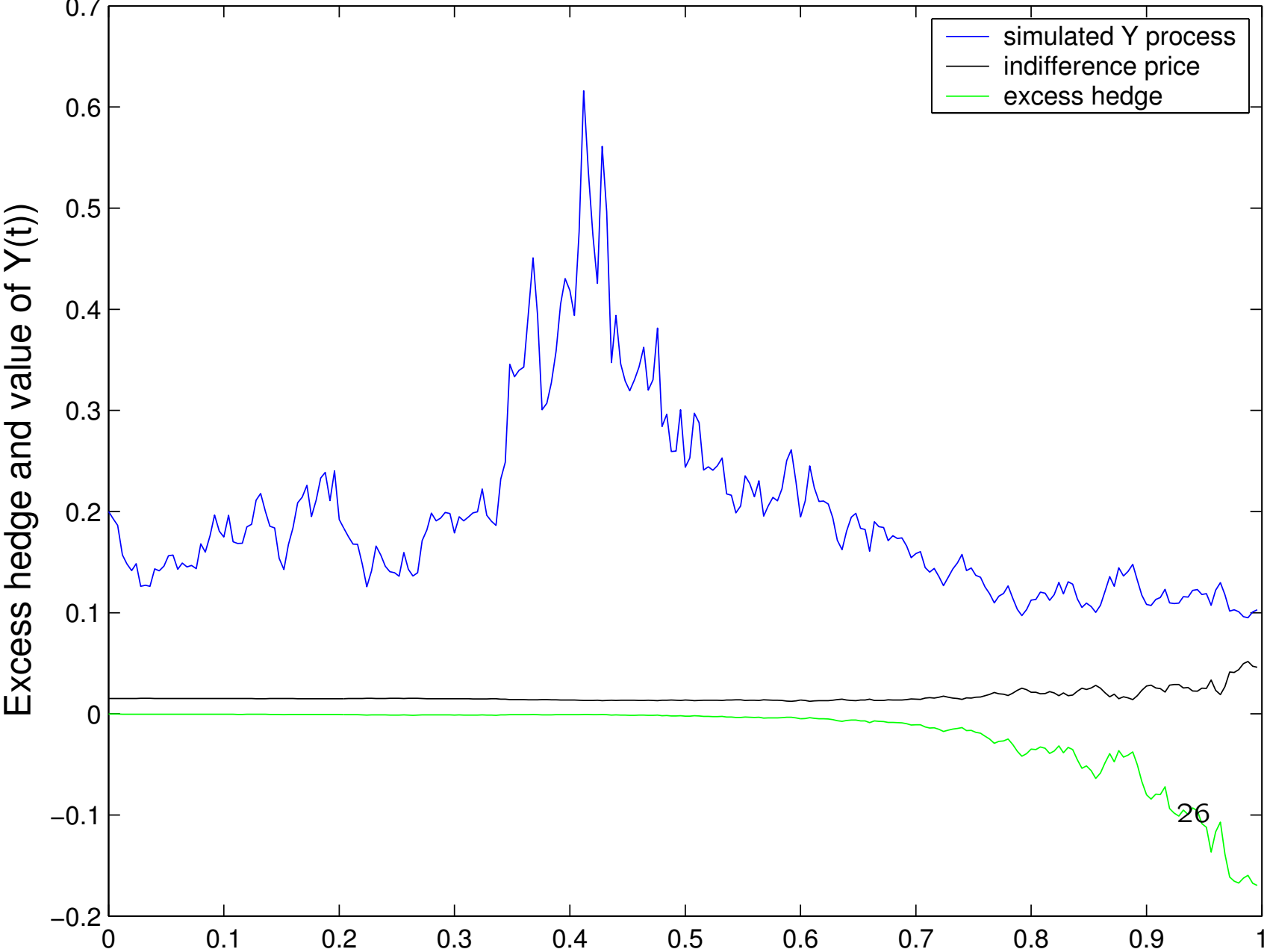
Volatility put versus $\log \gamma$ and Y_0



Volatility put versus $\log \gamma$ and Y_0

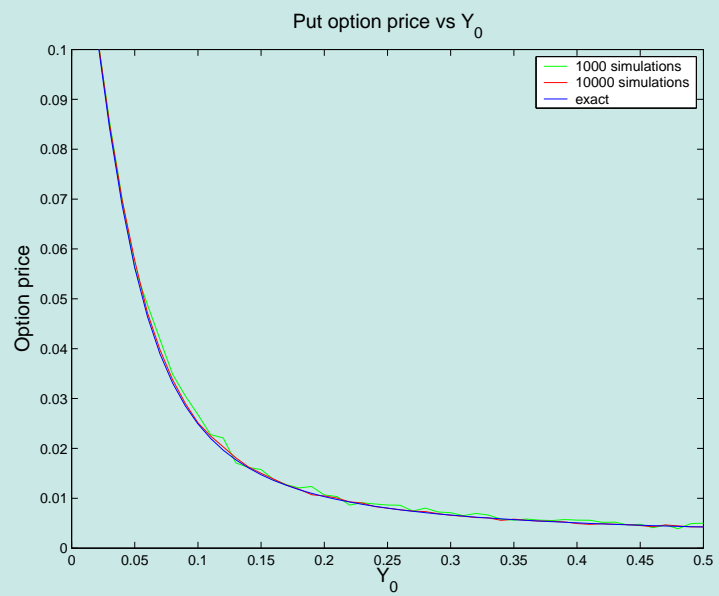


sample hedge process over one year



We now run the algorithm with the same model parameters as before (in particular $\gamma = 1$). To account for the portfolio dependence in both S_t and Y_t we took $\mathcal{R}(S)$ to be the six-dimensional space spanned by the functions $\{1, y, y^2, s, sy, s^2\}$.

We first applied the allocation algorithm to a volatility put option with payoff $(0.15 - \sigma_T^2)^+$ and time to maturity at $T = 0.2$ and computed the indifference prices with Y_0 varying in the interval $[0, 0.5]$.



Next we consider a put option on the stock, that is, with payoff $(K - S_T)^+$. The following pictures show the indifference prices and implied volatility surface with $N = 10000$ simulations.

